Journal of Applied Logic 10 (2012) 219–237

Contents lists available at SciVerse ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

Temporal alethic–deontic logic and semantic tableaux

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ARTICLE INFO

Article history:
Received 2 June 2011
Accepted 7 March 2012
Available online 15 March 2012

Keywords:
T × W logics
Temporal logic
Modal logic
Deontic logic
Semantic tableaux
Historical necessity
The ought-implies-can principle
The means-end principle

ABSTRACT

The purpose of this paper is to describe a set of temporal alethic–deontic systems, i.e. systems that include temporal, alethic and deontic operators. All in all we will consider 2,147,483,648 systems. All systems are described both semantically and proof theoretically. We use a kind of possible world semantics, inspired by the so-called T × W semantics, to characterize our systems semantically and semantic tableaux to characterize them proof theoretically. We also show that all systems are sound and complete with respect to their semantics.

1. Introduction

The purpose of this paper is to describe a set of temporal alethic–deontic systems, i.e. systems that include temporal, alethic and deontic operators. All systems are described both semantically and proof theoretically. We use a kind of possible world semantics, inspired by the so-called T × W semantics (see Section 3), to characterize our systems semantically and semantic tableaux (see Section 4) to characterize them proof theoretically. We also show that all systems are sound and complete with respect to their semantics.

Several philosophers and logicians have developed logical systems that deal with various combinations of conditions governing temporal, alethic and deontic elements (e.g. Chellas [15], Bailhache [5–8], van Eck [21], Thomason [38,39], Åqvist and Hoepelman [4], Åqvist [3], Bartha [9], Horty [28], Belnap, Perloff and Xu [10], Brown [11–13]). These are important contributions to deontic logic, since temporal, modal and deontic concepts seem to interact in a number of different interesting ways. There are many similarities but also some differences between these approaches and the current work. Various thinkers introduce different languages, different proof methods and different semantics. Most temporal alethic–deontic logicians use some kind of tree-like structure to describe their systems semantically, for instance the so-called Ockhamist frames perhaps first hinted at in Prior [35], and they try to find different axioms that correspond to different conditions that may be imposed on these structures.

In a number of essays Zanardo and co-workers have developed systems that combine modal and temporal logic (e.g. Zanardo [42], DiMaio and Zanardo [20], Ciuni and Zanardo [18]). Even though the systems described in these essays do not include a deontic part, the work is relevant for anyone interested in developing a temporal alethic–deontic logic. (See also von Kutschera [31], Åqvist [2] and Wölfl [41].)

If we may simplify matters, it seems to us that there are basically three kinds of semantics that have been used by temporal alethic–deontic logicians, T × W semantics (e.g. [3–8,15,40]), moment based (branching time) semantics (e.g. [9,10,28])

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1570-8683/$ – see front matter
http://dx.doi.org/10.1016/j.jal.2012.03.002

and branch based semantics (e.g. [11,12]). On the T × W approach, both times and worlds are basic and truth is relativized to both together. On the moment based (branching time) approach a set of concrete moments and a causal relation that orders the moments into a tree-like structure are basic. The past at any moment is usually taken to be fixed, while the future is open. A maximal set of linearly ordered moments is a history. Sentences are evaluated at moment/history pairs. According to the branch based semantics, branches are basic. We can say that two branches are related to each other when they have the same past and a moment can be viewed as an equivalence class under this alternative relation. Sentences are then evaluated at branches.

However, the differences between these types of semantics are not great and it is not clear how to classify various thinkers. Both Chellas [15] and Åqvist and Hoepelman [4], for instance, suggest that possible worlds might be interpreted as functions from a set of moments of time into a set of events or concrete situations. A possible world can then be seen as a possible course of events, a possible history or a temporally ordered sequence of events. So, their approaches are similar to the moment based (branching time) semantics. And e.g. Zanardo's work (see references above) shows that there are important connections between branch based semantics and T × W semantics. We use a kind of T × W semantics in this essay.

However, from a proof theoretical viewpoint, all of these earlier approaches to temporal alethic–deontic logic are axiomatic. As far as we know, no one has developed any semantic tableau systems that include temporal, alethic and deontic components. We are only aware of tableau systems for mono- and multi-modal systems that evaluate sentences at a single point (world, time) (see Section 4 for some relevant references). In all systems discussed in this essay truth is relativized to both possible worlds and moments in time. This requires a whole new set a tableau rules and new soundness and completeness proofs.

Why should we study these systems? We will briefly mention some considerations that seem to us to make this investigation both philosophically and logically interesting.

(i) Our temporal alethic–deontic systems are more expressive than various mono-modal systems that only include one kind of modal operator. We can therefore symbolize many sentences and arguments formulated in natural languages that cannot be symbolized in a mono-modal system.

Consider, for instance, the ought-implies-can principle (only what is possible is obligatory) and the means-end principle (every necessary consequence of what ought to be ought to be), which include both alethic and deontic concepts. It seems impossible to symbolize such principles using only one kind of modality, but we can formalize some versions of them in our systems (see Section 5).

We will now consider an argument that includes temporal, alethic and deontic concepts that seems clearly valid given some natural readings. Let us call this argument the *You ought to love your children* argument. We will mention two interpretations of this argument. According to the first, the concept of necessity in the second premise should be analyzed as universal or absolute necessity and according to the second, it should be analyzed as historical necessity (see Section 2.2). In Example 7 we will prove that the conclusion in the *You ought to love your children* argument is indeed derivable from the premises in the weakest tableau system T described in this essay (see Section 4.3), if the concept of necessity in the second premise is interpreted as absolute necessity. Since this system is sound with respect to the class of all models (see Sections 3 and 6), the argument is valid on this class given the first interpretation. This will illustrate how to use our systems to establish that an argument is valid. We will also show that the conclusion in the argument isn't derivable from the premises in T given that the concept of necessity is interpreted as historical necessity (see Example 8). Since T is complete with respect to the class of all models (see Section 6), it follows that the argument isn't valid on this class given the second interpretation. This will illustrate how we can use the tableau method to produce countermodels and show that an argument is invalid. We hope that this example will be enough to convince the reader that our systems have interesting philosophical applications and are therefore worth considering.

The *You ought to love your children* argument

It is always going to be the case that you ought to love your children.

It is necessary that if you love your children, then you respect your children and care for your children.

Hence, it is always going to be the case that you ought to respect your children and it is always going to be the case that you ought to care for your children.

We don't know whether the premises in this argument are (settled) true or not but we think that they are intuitively plausible and the argument seems clearly valid given certain natural interpretations of the second premise. Furthermore, we seem to need a temporal alethic–deontic logic to show this. In [24] Erich Fromm suggests that it is a conceptual or essential truth that (true) love includes respect and caring (among other things). If this is correct, it seems reasonable to claim that it is (absolutely) necessary that you (truly) love someone only if you respect this person and care for this person. Furthermore, it appears to be a reasonable norm that (it is settled that) it is always going to be the case that you ought to love your children (at least for most parents as long as they live). So, both premises seem plausible, or at least interesting. This and countless other arguments suggest that we need temporal alethic–deontic logic.

Of course, as one of the anonymous referees pointed out, the fact that multi-modal logics are more expressive than mono-modal systems comes with a price in terms of computational complexity. For some purposes, it may be desirable to develop logical systems that are as simple as possible, for instance, if we want to implement our systems on a computer. But if we are interested in using our logics to analyze sentences and arguments formulated in natural languages that include temporal, alethic and deontic concepts we believe that we need very rich systems.

(ii) It might be possible to use some of our systems to solve at least some deontic paradoxes, e.g. Priors' paradoxes of derived obligation (see Prior [34]) and Chisholms' contrary-to-duty paradox (see Chisholm [17]) that are problematic for pure monadic deontic systems (see e.g. Bailhache [6], which suggests that a system similar to our aTB4dD45adMOOC45t4FCBCadtSPSRPIFTBT can be used to solve these paradoxes). However, to be able to symbolize contrary-to-duty obligations we probably need to add some kind of dyadic deontic operator to our systems (in [7] Bailhache seems to come to this conclusion too). Hopefully, we will be able to show how dyadic deontic logic can be combined with the systems in this essay in later work.

(iii) We can use our systems to analyze or elucidate some interesting concepts that have been discussed by moral philosophers and deontic logicians for some time, e.g. the concept of a prima facie obligation. (See van Eck [21] (especially Chapters II and IV) for an idea about how to do this.)

(iv) In some of our systems we can prove that the so-called wide and narrow conditional obligation sentences are equivalent under certain conditions (see Section 5, Theorem 4 and Table 23). These equivalences can be used to shed some light on the so-called dilemma of commitment and detachment (see van Eck [21] especially Chapters II and IV).

(v) Many people think that a main function of norms and deontic sentences is to guide human behaviour and that only future actions can be guided. This idea can be given an interesting interpretation in our systems (see Section 5).

(vi) Semantic tableau systems are often more user-friendly than corresponding axiomatic systems. It is often easier to prove sentences, to check whether a set of sentences is consistent, to decide the validity of an argument, etc. in a semantic tableau system. Furthermore, it is often relatively easy to prove things about various systems of this kind. This is a good reason not to focus exclusively on Hilbert-style temporal alethic–deontic logics as deontic logicians have tended to do up until now.

(vii) Some temporal alethic–deontic systems discussed in the literature have only been proved sound with respect to their semantics, e.g. the system DARB constructed by Åqvist and Hoepelman (see [4]). (DARB is probably incomplete, as pointed out by Bailhache [7].) We prove that all our systems are both sound and complete. Furthermore, we show that all our systems are sound and complete in a uniform way, using the same method for all systems.

(viii) Choices between different temporal, alethic and deontic systems can to a large extent be made independently, and we can combine our semantic conditions and tableau rules in many different ways. All such combinations lead to a total of 2,147,483,648 systems. Many of these are deductively equivalent, but many are also non-equivalent. We think that it is a good thing that we have many systems to choose from. Some of our logics are perhaps not philosophically relevant or interesting, but it seems likely that at least some, perhaps many, of them are. Different systems may perhaps be used for different purposes and to symbolize different alethic, temporal and deontic concepts.

As an anonymous referee correctly pointed out, one cannot derive the fact that (it seems likely that) at least some of the logics are philosophically relevant or interesting from their sheer number. But of course we are not claiming that. The reasons why this seems likely to us are much more complicated. It seems likely because most of our semantic conditions and tableau rules seem plausible in themselves and because our temporal alethic–deontic tableau systems include many mono-modal temporal, alethic and deontic systems that appear interesting and that are well-known in the literature and that many philosophers and logicians have considered plausible. It seems likely because our systems are consistent and contain theorems that seem intuitively attractive to us. It seems likely because the systems cohere well with our belief sets. It seems likely because even though many of the systems seem interesting, some of them include theorems that are controversial and that some philosophers may want to deny. E.g. the ought-implies-can principle appears reasonable if "ought" is interpreted as a moral notion or "ought all-things considered", but perhaps there are interpretations of the concept on which this principle is not plausible, for instance if "ought" is interpreted as "prima facie ought" or "ought according to the law". So, for some applications we may want to include the rules (*T-OC*) or (*T-OC′*) specified below in our tableau system and for other applications we may want to exclude them. And so on.

(ix) By imposing different semantic conditions and adding appropriate tableau rules in an obvious way, our systems can be made to include many of the standard, normal alethic, deontic and temporal systems that can be found in the literature (e.g. M, B, $S4$, $S5$, etc.). (See e.g. [1,14,16,25,27,36] and the introduction to [32] for more on some basic alethic, deontic and temporal systems.)

(x) We consider some semantic conditions and theorems that have not been discussed before in connection with temporal alethic–deontic logic (e.g. *C-WPI*, *C-OC′*, *C-MO′*, *C-ab*5, *C-PMP*, *C-OMP*, *C-MOP*, etc.), at least not explicitly and as far as we are aware. Yet they seem philosophically and/or theoretically interesting.

The essay is divided into 6 sections. In Section 2 we describe the syntax of our systems and in Section 3 their semantics. Section 4 deals with the proof theoretic characterization of our logics and Section 5 includes some examples of theorems. Finally, Section 6 contains soundness and completeness proofs for every system.

2. Syntax

2.1. Alphabet

(i) A denumerably infinite set Prop of proposition letters p, q, r, s, p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, $\ldots$, (ii) a denumerably infinite set NT of names of times $t0$, $t1$, $t2$, $t3$, $\ldots$, (iii) the primitive truth-functional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication) and $\leftrightarrow$ (material equivalence), (iv) the alethic operators U, M, $\Box$ and $\Diamond$, (v) the temporal operators R, $\underline{G}$, $\underline{H}$, $\underline{F}$ and $\underline{P}$, (vi) the deontic operators O and P, and (vii) the brackets (,).

2.2. Language

The language L is the set of well-formed formulas (wffs) generated by the usual clauses for proposition letters and propositionally compound sentences, and the following clauses: (i) if A is a wff, then UA ("it is universally (or absolutely) necessary that A"), MA ("it is universally (or absolutely) possible that A"), $\Box A$ ("it is historically necessary (or settled) that A"), $\Diamond A$ ("it is historically possible that A"), $\underline{G}A$ ("it is always going to be the case that A"), $\underline{H}A$ ("it has always been the case that A"), $\underline{F}A$ ("it will some time in the future be the case that A"), $\underline{P}A$ ("it was some time in the past the case that A"), OA ("it ought to be the case that A") and PA ("it is permitted that A") are wffs, (ii) if A is a wff and t is in NT, then RtA ("it is realized at time t that A") is a wff, and (iii) nothing else is a wff.

Capital letters A, B, C, $\ldots$ are used to represent arbitrary (not necessarily atomic) formulas of the object language. The upper case Greek letter Γ represents an arbitrary set of formulas. Brackets around sentences are usually dropped if the result is not ambiguous.

2.3. Definitions

$\not\Diamond A$ ("it is impossible that A") $= \neg\Diamond A$, FA ("it is forbidden that A") $= \neg PA$, ∇A ("it is historically contingent that A") $= \Diamond A \wedge \Diamond\neg A$, $\triangle A$ ("it is historically non-contingent that A") $= \neg\nabla A$ (or $\Box A \vee \Box\neg A$), $\underline{A}A$ ("it is always true that A") $= \underline{H}A \wedge A \wedge \underline{G}A$, $\underline{S}A$ ("sometimes it is true that A") $= \neg\underline{A}\neg A$ (or $\underline{P}A \vee A \vee \underline{F}A$), $[G]A = A \wedge \underline{G}A$, $\langle F\rangle A = \neg[G]\neg A$ (or $A \vee \underline{F}A$), $[H]A = A \wedge \underline{H}A$, $\langle P\rangle A = \neg[H]\neg A$ (or $A \vee \underline{P}A$), $A \Rightarrow B = \Box(A \rightarrow B)$, $A \Leftrightarrow B = \Box(A \leftrightarrow B)$.

3. Semantics

3.1. Basic concepts

3.1.1. Temporal alethic–deontic frame

A temporal alethic–deontic frame F is a relational structure $\langle W, T, <, R, S\rangle$, where W is a non-empty set of possible worlds, T is a non-empty set of times, $<$ is a binary relation on T ($< \subseteq T \times T$) and R and S are two ternary accessibility relations ($R \subseteq W \times W \times T$ and $S \subseteq W \times W \times T$).

R "corresponds" to the alethic operators $\Box$ and $\Diamond$, $<$ to the temporal operators $\underline{G}$, $\underline{F}$, $\underline{H}$ and $\underline{P}$ and S to the deontic operators O and P. Informally, $\tau < \tau'$ says that the time τ is before the time τ' (or that τ' is later than τ) $R\omega\omega'\tau$ says that the possible world ω' is alethically accessible from the possible world ω at time τ, and $S\omega\omega'\tau$ says that ω' is deontically accessible from ω at τ.

3.1.2. Temporal alethic–deontic model

A temporal alethic–deontic model M is a triple $\langle F, V, v\rangle$ where: (i) F is a temporal alethic–deontic frame; (ii) V is a valuation or interpretation function, which to every proposition letter p in Prop assigns a subset of $W \times T$, i.e. a set of ordered pairs $\langle\omega, \tau\rangle$, where $\omega \in W$ and $\tau \in T$; and (iii) v is a function which to each temporal name in NT assigns a time in T.

When $M = \langle F, V, v\rangle$ we say that M is based on the frame F, or that F is the frame underlying M. To save space, we shall also use the following notation for a temporal alethic–deontic model: $\langle W, T, <, R, S, V, v\rangle$, where W, T, $<$, R, S, V and v are interpreted as usual. "**F**" stands for a class of frames and "**M**" for a class of models.

3.1.3. Truth in a model

Let M be any temporal alethic–deontic model $\langle F, V, v\rangle$, based on a temporal alethic–deontic frame $F = \langle W, T, <, R, S\rangle$. Let $\omega \in W$, $\tau \in T$ and let A be a well-formed sentence in L. $M, \omega, \tau \Vdash A$ abbreviates A is true at or in the possible world ω at the time τ in the temporal alethic–deontic model M (or A is true at the pair $\langle\omega, \tau\rangle$ in M). The truth conditions for proposition letters and complex sentences are given in the following list. Those for truth-functional connectives are the usual ones (illustrated by conjunction):

(i)	$M, \omega, \tau \Vdash p$	iff	$\langle \omega, \tau \rangle \in V(p)$ for any p in Prop,
(ii)	$M, \omega, \tau \Vdash A \wedge B$	iff	$M, \omega, \tau \Vdash A$ and $M, \omega, \tau \Vdash B$,
(iii)	$M, \omega, \tau \Vdash \Box A$	iff	$\forall \omega' \in W$ s.t. $R\omega\omega'\tau$: $M, \omega', \tau \Vdash A$,
(iv)	$M, \omega, \tau \Vdash \Diamond A$	iff	$\exists \omega' \in W$ s.t. $R\omega\omega'\tau$: $M, \omega', \tau \Vdash A$,
(v)	$M, \omega, \tau \Vdash \underline{G}A$	iff	$\forall \tau' \in T$ s.t. $\tau < \tau'$: $M, \omega, \tau' \Vdash A$,
(vi)	$M, \omega, \tau \Vdash \underline{F}A$	iff	$\exists \tau' \in T$ s.t. $\tau < \tau'$: $M, \omega, \tau' \Vdash A$,
(vii)	$M, \omega, \tau \Vdash \underline{H}A$	iff	$\forall \tau' \in T$ s.t. $\tau' < \tau$: $M, \omega, \tau' \Vdash A$,
(viii)	$M, \omega, \tau \Vdash \underline{P}A$	iff	$\exists \tau' \in T$ s.t. $\tau' < \tau$: $M, \omega, \tau' \Vdash A$,
(ix)	$M, \omega, \tau \Vdash Rt'A$	iff	$M, \omega, v(t') \Vdash A$, for all $t' \in NT$,
(x)	$M, \omega, \tau \Vdash OA$	iff	$\forall \omega' \in W$ s.t. $S\omega\omega'\tau$: $M, \omega', \tau \Vdash A$,
(xi)	$M, \omega, \tau \Vdash PA$	iff	$\exists \omega' \in W$ s.t. $S\omega\omega'\tau$: $M, \omega', \tau \Vdash A$,
(xii)	$M, \omega, \tau \Vdash UA$	iff	$\forall \omega' \in W$ and $\forall \tau' \in T$: $M, \omega', \tau' \Vdash A$,
(xiii)	$M, \omega, \tau \Vdash MA$	iff	$\exists \omega' \in W$ and $\exists \tau' \in T$: $M, \omega', \tau' \Vdash A$.

3.1.4. Validity, satisfiability, logical consequence, etc.

Now we are in a position to define several important semantic concepts.

Validity in a class of models. A sentence A is valid on or in a class of models $\mathbf{M}$ ($\mathbf{M} \Vdash A$) iff A is true at every pair $\langle \omega, \tau \rangle$ in every model in $\mathbf{M}$.

Satisfiability in a class of models. A set of sentences Γ is satisfiable in a class of models $\mathbf{M}$ iff every sentence in Γ is true at some pair $\langle \omega, \tau \rangle$ in some model in $\mathbf{M}$.

Logical consequence in a class of models. A sentence B is a logical consequence of a set of sentences Γ on or in a class of models $\mathbf{M}$ ($\mathbf{M}, \Gamma \Vdash B$) iff B is true at every pair $\langle \omega, \tau \rangle$ in every model in $\mathbf{M}$ at which all members of Γ are true.

The concepts of validity, satisfiability and logical consequence in a class of frames are defined similarly.

3.2. Conditions on frames and models

We will explore several different conditions on our frames and models in this section. The conditions are divided into six classes. The first class tells us something about the formal properties of the relation $<$, the second about the formal properties of the relation R (at a time), the third about the formal properties of the relation S (at a time), the fourth about how R and S are related to each other (at a time), the fifth about how R and S are related to $<$, and the sixth consists of two conditions that we may impose on the valuation function V in a model.

The variables x, y, z, w in Tables 1–6 are taken to range over possible worlds in W, t, t', t'' over times in T, and the symbols $\wedge$, $\rightarrow$, $\forall$ and $\exists$ are used as metalogical symbols in the standard way. Let $F = \langle W, T, <, R, S \rangle$ be a temporal alethic–deontic frame and $M = \langle W, T, <, R, S, V, v \rangle$ be a temporal alethic–deontic model. If $\forall t \forall x \exists y Sxyt$, we say that S satisfies or fulfills condition C-dD and also that F and M satisfy or fulfill condition C-dD and similarly in all other cases. C-dD is called "C-dD" because the tableau rule T-dD "corresponds" to C-dD and the sentence dD is valid on the class of all frames and in the class of all models that satisfy condition C-dD and similarly in all other cases. Let C be any of the conditions in Tables 1–6. Then a C-frame is a frame that satisfies condition C and a C-model is a model that satisfies C.

Most of these conditions are self-explanatory. Nevertheless, we will add a few comments.

The conditions on $<$ are well-known from temporal logic (see e.g. [36] and [14]). Intuitively, C-PD (as in Past D) says that there is no first point in time and C-FD (as in Future D) that there is no last point in time. C-$t4$ (as in temporal 4) claims that time is transitive and C-DE (as in DEnse) that time is dense. C-PC (as in Past Convergence) says that time doesn't branch towards the past and C-FC (as in Future Convergence) that time doesn't branch towards the future.

The conditions on R and S are similar to well-known conditions on the alethic and deontic accessibility relations in mono-modal alethic and deontic logic, respectively (see e.g. [1,16]). The only difference is that R and S are 3-place relations in our systems. Intuitively, this corresponds to the idea that the ordinary 2-place relations R and S are relativized to particular moments in time. So, intuitively C-aT says that R is reflexive at every time and C-dD says that S is serial at every time etc.

The conditions concerning the relation between R and S correspond to interesting bi-modal principles. The sentence MO (as in the Must-Ought principle) is valid on the class of frames that satisfies C-MO. The sentence OC (as in the Ought-Can principle) is valid on the class of frames that satisfies C-OC etc. "PMP" stands for "Permitted-Must Permutation", "OMP" stands for "Ought-Must Permutation" and "MOP" stands for "Must-Ought Permutation". (In two (currently) unpublished essays *Bimodal Logic* and *Bimodal Logic and Semantic Tableaux* we investigate these principles more closely. For some general ideas about how to combine two or more modal systems, see e.g. Kracht [30] and Gabbay, Kurucz, Wolter and Zakharyaschev [26].)

Intuitively, C-SP (as in Shared Past) says that if a world y is alethically accessible from a world x at a certain time t' then y is also alethically accessible from x at all earlier times t. C-SR (as in Secondary Ramification) says that if a time t is before a time t' and the world y is deontically accessible from the world x at t and the world z is deontically accessible from y at t', then z is also deontically accessible from y at t. According to C-PI (as in Post-Implication) the world z is deontically accessible from the world y at time t' if y is deontically accessible from the world x at time t, z is alethically accessible from y at t' and t is before t'. C-WPI (as in Weak Post-Implication) is interpreted similarly. A brief discussion of

$C\text{-}SR$, $C\text{-}PI$ and $C\text{-}WPI$ is included in Section 5. $C\text{-}FT$ (as in Forward Transfer) says that if p is true at world x at a time t and world y is alethically accessible from x at t, then p is true in y at t. $C\text{-}BT$ (as in Backward Transfer) is interpreted similarly. Note that p in these conditions is atomic. Together $C\text{-}FT$ and $C\text{-}BT$ say that if world y is alethically accessible from world x at time t, then p is true in x at t iff p is true in y at t, for every atomic sentence p. $C\text{-}SP$, $C\text{-}SR$ and $C\text{-}PI$ are mentioned by Bailhache in several works, e.g. in [6–8]. $C\text{-}WPI$ is a weaker condition than $C\text{-}PI$ that we find intuitively more plausible.

3.2.1. Conditions on the relation $<$

Table 1

Condition	Formalization of condition
$C\text{-}PD$	$\forall t \exists t' t' < t$
$C\text{-}FD$	$\forall t \exists t' t < t'$
$C\text{-}t4$	$\forall t \forall t' \forall t''((t < t' \wedge t' < t'') \rightarrow t < t'')$
$C\text{-}DE$	$\forall t \forall t'(t < t' \rightarrow \exists t''(t < t'' \wedge t'' < t'))$
$C\text{-}FC$	$\forall t \forall t' \forall t''((t < t' \wedge t < t'') \rightarrow (t' < t'' \vee t' = t'' \vee t'' < t'))$
$C\text{-}PC$	$\forall t \forall t' \forall t''((t' < t \wedge t'' < t) \rightarrow (t' < t'' \vee t' = t'' \vee t'' < t'))$

3.2.2. Conditions on the relation R

Table 2

Condition	Formalization of condition
$C\text{-}aT$	$\forall t \forall x Rxxt$
$C\text{-}aD$	$\forall t \forall x \exists y Rxyt$
$C\text{-}aB$	$\forall t \forall x \forall y(Rxyt \rightarrow Ryxt)$
$C\text{-}a4$	$\forall t \forall x \forall y \forall z((Rxyt \wedge Ryzt) \rightarrow Rxzt)$
$C\text{-}a5$	$\forall t \forall x \forall y \forall z((Rxyt \wedge Rxzt) \rightarrow Ryzt)$

3.2.3. Conditions on the relation S

Table 3

Condition	Formalization of condition
$C\text{-}dD$	$\forall t \forall x \exists y Sxyt$
$C\text{-}d4$	$\forall t \forall x \forall y \forall z((Sxyt \wedge Syzt) \rightarrow Sxzt)$
$C\text{-}d5$	$\forall t \forall x \forall y \forall z((Sxyt \wedge Sxzt) \rightarrow Syzt)$
$C\text{-}dT'$	$\forall t \forall x \forall y(Sxyt \rightarrow Syyt)$
$C\text{-}dB'$	$\forall t \forall x \forall y \forall z((Sxyt \wedge Syzt) \rightarrow Szyt)$

3.2.4. Conditions concerning the relation between R and S

Table 4

Condition	Formalization of condition
$C\text{-}MO$	$\forall t \forall x \forall y(Sxyt \rightarrow Rxyt)$
$C\text{-}OC$	$\forall t \forall x \exists y(Sxyt \wedge Rxyt)$
$C\text{-}OC'$	$\forall t \forall x \forall y(Sxyt \rightarrow \exists z(Ryzt \wedge Syzt))$
$C\text{-}MO'$	$\forall t \forall x \forall y \forall z((Sxyt \wedge Syzt) \rightarrow Ryzt)$
$C\text{-}ad4$	$\forall t \forall x \forall y \forall z((Rxyt \wedge Syzt) \rightarrow Sxzt)$
$C\text{-}ad5$	$\forall t \forall x \forall y \forall z((Rxyt \wedge Sxzt) \rightarrow Syzt)$
$C\text{-}PMP$	$\forall t \forall x \forall y \forall z((Sxyt \wedge Rxzt) \rightarrow \exists w(Rywt \wedge Szwt))$
$C\text{-}OMP$	$\forall t \forall x \forall y \forall z((Rxyt \wedge Syzt) \rightarrow \exists w(Sxwt \wedge Rwzt))$
$C\text{-}MOP$	$\forall t \forall x \forall y \forall z((Sxyt \wedge Ryzt) \rightarrow \exists w(Rxwt \wedge Swzt))$

3.2.5. Conditions concerning the relation between R, S and $<$

Table 5

Condition	Formalization of condition
$C\text{-}SP$	$\forall t \forall t' \forall x \forall y((t < t' \wedge Rxyt') \rightarrow Rxyt)$
$C\text{-}SR$	$\forall t \forall t' \forall x \forall y \forall z((t < t' \wedge Sxyt \wedge Syzt') \rightarrow Syzt)$
$C\text{-}WPI$	$\forall t \forall t' \forall x \forall y((t < t' \wedge Sxyt \wedge Rxyt') \rightarrow Sxyt')$
$C\text{-}PI$	$\forall t \forall t' \forall x \forall y \forall z((t < t' \wedge Sxyt \wedge Ryzt') \rightarrow Syzt')$

3.2.6. Conditions on the valuation function V in a model

Table 6

Condition	Formalization of condition
$C\text{-}FT$	If $Rxyt$ and $\langle x, t\rangle \in V(p)$, then $\langle y, t\rangle \in V(p)$ for all p in Prop, t in T and x and y in W
$C\text{-}BT$	If $Rxyt$ and $\langle y, t\rangle \in V(p)$, then $\langle x, t\rangle \in V(p)$ for all p in Prop, t in T and x and y in W

3.3. Classification of model classes

The 31 conditions on our models listed in Tables 1–6 can be used to obtain a categorization of the set of all models into various kinds. All in all there are 2,147,483,648 different combinations of these conditions. In general, we shall say that $\mathbf{M}(C_1, \ldots, C_n)$ is the class of (all) models that satisfies the conditions $C_1, \ldots, C_n$. E.g. $\mathbf{M}(C\text{–}dD, C\text{–}aT, C\text{–}MO)$ is the class of all models that satisfies $C\text{-}dD$, $C\text{-}aT$ and $C\text{-}MO$.

3.4. The logic of a class of models

The set of all sentences (in L) that are valid in a class of models $\mathbf{M}$ is called the logical system of (the system of or the logic of) $\mathbf{M}$, in symbols $S(\mathbf{M}) = \{A \in L\colon\ \mathbf{M} \Vdash A\}$. E.g. $S(\mathbf{M}(C\text{–}dD, C\text{–}aT, C\text{–}MO))$ is the set of all sentences that are valid in the class of all models that satisfies $C\text{-}dD$, $C\text{-}aT$ and $C\text{-}MO$.

By using the classification of model classes mentioned in Section 3.3 we obtain 2,147,483,648 different systems defined in this way. In the next section we will develop a set of semantic tableau systems that exactly correspond to these logics.

4. Proof theory

4.1. Semantic tableaux

The kind of semantic tableau systems we use is inspired by Melvin Fitting and Graham Priest (see e.g. Fitting [22], Fitting and Mendelsohn [23] and Priest [33]). The propositional part is similar to systems introduced by Raymond Smullyan [37] and Richard Jeffrey [29].

The concepts of semantic tableau, branch, open and closed branch, etc. are essentially defined as in Priest [33]. However, there are some minor differences. A node in a tree now has one of the following forms: $ti < tj$ (time $\tau[i]$ is before time $\tau[j]$), $rwiwjtk$ (world ωj is alethically accessible from world ωi at time $\tau[k]$), $swiwjtk$ (world ωj is deontically accessible from world ωi at time $\tau[k]$), $ti = tj$ (the time $\tau[i]$ is the same as the time $\tau[j]$), or $A, witj$ (A is true in world ωi at time $\tau[j]$) where A is a wff in L. A tableau branch is closed iff it contains anything of the form $A, witj$ and $\neg A, witj$. ($\tau[i]$ is the time picked out by ti, ωj the world picked out by wj, etc.) For more on semantic tableaux, see D'Agostino, Gabbay, Hähnle and Posegga [19].

4.2. Tableau rules

4.2.1. Propositional rules

We use the same propositional rules as in Priest [33] modified in an obvious way. We call them $(\wedge)$, $(\neg\wedge)$, etc.

4.2.2. Basic alethic rules (ba-rules)

Table 7

U	M	$\Box$	$\Diamond$
$UA, witj$ $\downarrow$ $A, wktl$ for any wk and tl	$MA, witj$ $\downarrow$ $A, wktl$ where wk and tl are new	$\Box A, witk$ $rwiwjtk$ $\downarrow$ $A, wjtk$	$\Diamond A, witk$ $\downarrow$ $rwiwjtk$ $A, wjtk$ where wj is new
$\neg U$	$\neg M$	$\neg\Box$	$\neg\Diamond$
$\neg UA, witj$ $\downarrow$ $M\neg A, witj$	$\neg MA, witj$ $\downarrow$ $U\neg A, witj$	$\neg\Box A, witj$ $\downarrow$ $\Diamond\neg A, witj$	$\neg\Diamond A, witj$ $\downarrow$ $\Box\neg A, witj$

4.2.3. Basic deontic rules (bd-rules)

The basic d-rules look exactly like the basic a-rules for $\Box$, $\Diamond$, $\neg\Box$, $\neg\Diamond$, except that $\Box$ is replaced by O, $\Diamond$ by P, and r by s. We give them similar names.

4.2.4. Basic temporal rules (bt-rules), Id(I) and Id(II)

Table 8

$\underline{G}$	$\underline{H}$	$\underline{F}$	$\underline{P}$
$\underline{G}A, witj$ $tj < tk$ $\downarrow$ $A, witk$	$\underline{H}A, witj$ $tk < tj$ $\downarrow$ $A, witk$	$\underline{F}A, witj$ $\downarrow$ $tj < tk$ $A, witk$ where tk is new	$\underline{P}A, witj$ $\downarrow$ $tk < tj$ $A, witk$ where tk is new
$\neg\underline{G}$	$\neg\underline{H}$	$\neg\underline{F}$	$\neg\underline{P}$
$\neg\underline{G}A, witj$ $\downarrow$ $\underline{F}\neg A, witj$	$\neg\underline{H}A, witj$ $\downarrow$ $\underline{P}\neg A, witj$	$\neg\underline{F}A, witj$ $\downarrow$ $\underline{G}\neg A, witj$	$\neg\underline{P}A, witj$ $\downarrow$ $\underline{H}\neg A, witj$

Table 9

Rt	$\neg Rt$	$Id(I)$	$Id(II)$
$RtiA, wjtk$ $\downarrow$ $A, wjti$	$\neg RtiA, wjtk$ $\downarrow$ $Rti\neg A, wjtk$	$A(ti)$ $ti = tj$ $\downarrow$ $A(tj)$	$A(ti)$ $tj = ti$ $\downarrow$ $A(tj)$

4.2.5. Alethic accessibility rules (a-rules)

Table 10

T-aD	T-aT	T-aB	T-$a4$	T-$a5$
$witk$ $\downarrow$ $rwiwjtk$ where wj is new	$witj$ $\downarrow$ $rwiwitj$	$rwiwjtk$ $\downarrow$ $rwjwitk$	$rwiwjtl$ $rwjwktl$ $\downarrow$ $rwiwktl$	$rwiwjtl$ $rwiwktl$ $\downarrow$ $rwjwktl$

4.2.6. Temporal accessibility rules (t-rules)

Table 11

T-$t4$	T-PD	T-FD
$ti < tj$ $tj < tk$ ↓ $ti < tk$	tj ↓ $tk < tj$ where tk is new	tj ↓ $tj < tk$ where tk is new
T-DE	T-FC	T-PC
$ti < tj$ ↓ $ti < tk$ $tk < tj$ where tk is new	$ti < tj$ $ti < tk$ ↙↓↘ $tj < tk$ $tj = tk$ $tk < tj$	$tj < ti$ $tk < ti$ ↙↓↘ $tj < tk$ $tj = tk$ $tk < tj$

4.2.7. Deontic accessibility rules (d-rules)

Table 12

T-dD	T-$d4$	T-$d5$	T-dT'	T-dB'
$witk$ ↓ $swiwjtk$ where wj is new	$swiwjtl$ $swjwktl$ ↓ $swiwktl$	$swiwjtl$ $swiwktl$ ↓ $swjwktl$	$swiwjtl$ ↓ $swjwjtl$	$swiwjtl$ $swjwktl$ ↓ $swkwjtl$

4.2.8. Alethic-deontic accessibility rules (ad-rules)

Table 13

T-MO	T-MO'	T-OC	T-OC'	
$swiwjtk$ ↓ $rwiwjtk$	$swiwjtl$ $swjwktl$ ↓ $rwjwktl$	$witk$ ↓ $swiwjtk$ $rwiwjtk$ where wj is new	$swiwjtl$ ↓ $rwjwktl$ $swjwktl$ where wk is new	
T-$ad4$	T-$ad5$	T-PMP	T-OMP	T-MOP
$rwiwjtl$ $swjwktl$ ↓ $swiwktl$	$rwiwjtl$ $swiwktl$ ↓ $swjwktl$	$swiwjtm$ $rwiwktm$ ↓ $rwjwltm$ $swkwltm$ where wl is new	$rwiwjtm$ $swjwktm$ ↓ $swiwltm$ $rwlwktm$ where wl is new	$swiwjtm$ $rwjwktm$ ↓ $rwiwltm$ $swlwktm$ where wl is new

4.2.9. Rules concerning R, S, $<$ and V (adt-rules)

Table 14

T-FT	T-BT	T-SP
$A, witk$ $rwiwjtk$ ↓ $A, wjtk$ where A is atomic	$A, wjtk$ $rwiwjtk$ ↓ $A, witk$ where A is atomic	$rwiwjtl$ $tk < tl$ ↓ $rwiwjtk$
T-SR	T-PI	T-WPI
$swiwjtl$ $tl < tm$ $swjwktm$ ↓ $swjwktl$	$swiwjtl$ $tl < tm$ $rwjwktm$ ↓ $swjwktm$	$swiwjtk$ $tk < tl$ $rwiwjtl$ ↓ $swiwjtl$

The temporal rules $Id(I)$ and $Id(II)$ may be interpreted in the following way. If $A(ti)$ is $A, wkti$, $A(tj)$ is $A, wktj$. If $A(ti)$ is $tk < ti$, $A(tj)$ is $tk < tj$. If $A(ti)$ is $ti = tk$, $A(tj)$ is $tj = tk$. If $A(ti)$ is $rwkwlti$, $A(tj)$ is $rwkwltj$. If $A(ti)$ is $swkwlti$, $A(tj)$ is $swkwltj$. There are infinitely many Rt and $\neg Rt$ rules, a pair for every t in NT.

4.3. Tableau systems

A tableau system is a set of tableau rules. A temporal alethic–deontic tableau system includes all propositional rules, all basic alethic rules, all basic deontic rules and all basic temporal rules (Sections 4.2.1–4.2.4, Tables 7–9). The minimal temporal alethic–deontic tableau system is called "T". $Id(I)$ and $Id(II)$ are added to every system that includes T-FC or T-PC (they are redundant in every other system). By adding any subset of the rules introduced in Sections 4.2.5–4.2.9 (Tables 10–14), we obtain an extension of T. Many of the 2,147,483,648 different systems obtained in this way are deductively equivalent, i.e. contain exactly the same set of theorems. We use the following conventions for naming systems. We write "$aA_1 \ldots A_i dB_1 \ldots B_j adC_1 \ldots C_k tD_1 \ldots D_l adtE_1 \ldots E_m$", where $A_1 \ldots A_i$ is a list (possibly empty) of a-rules, $B_1 \ldots B_j$ is a list (possibly empty) of d-rules, $C_1 \ldots C_k$ is a list (possibly empty) of ad-rules, $D_1 \ldots D_l$ is a list (possibly empty) of t-rules, and $E_1 \ldots E_m$ is a list (possibly empty) of adt-rules. We sometimes abbreviate by omitting "redundant" letters in a name, if it doesn't lead to any ambiguity. E.g. $aTdDadOCt4adtSP$ is the temporal alethic–deontic system that includes the rules T-aT, T-dD, T-OC, T-$t4$ and T-SP.

4.4. Proof, derivation, theorem, consistency in a system, etc.

The concepts of proof, theorem, derivation, consistency, inconsistency in a system etc. can essentially be defined in the usual way. Nevertheless, we will mention a few examples to show how to modify these definitions in the current setting.

Let S be a tableau system and let an S-tableau be a tableau generated in accordance with the rules in S.

Proof in a system. A proof of A in S is a closed S-tableau for $\neg A, w0t0$, i.e. a closed S-tableau whose root consists of $\neg A, w0t0$.

Theorem in a system. A is a theorem in S or provable in S or syntactically valid in S ($\vdash_S A$) iff there is a proof of A in S, i.e. iff there is a closed S-tableau for $\neg A, w0t0$.

Derivation in a system. A derivation or argument in the system S of B from the (finite) set of formulas Γ, is a closed S-tableau whose initial list comprises $A, w0t0$ for every $A \in \Gamma$ and $\neg B, w0t0$. The sentences A in Γ are called the premises of the derivation or argument and B is called the conclusion of the derivation or argument. The initial list of a tableau consists of the first nodes in this tableau whose satisfiability we are testing.

Proof-theoretic consequence in a system. B is a proof-theoretic consequence of the set of formulas Γ in S or B is derivable from a set of formulas Γ in S ($\Gamma \vdash_S B$) iff there is a derivation of B in S from Γ, i.e. iff there is a closed S-tableau whose initial list comprises $A, w0t0$ for every $A \in \Gamma$ and $\neg B, w0t0$. If an argument or inference consists of the premises Γ and the conclusion B, we shall say that this argument is (proof-theoretically) valid in a system S iff ($\Gamma \vdash_S B$), i.e. iff B is derivable from Γ in S.

Consistency and inconsistency. Γ is consistent in a system S iff it is not possible to derive a contradiction from Γ in S. Γ is inconsistent in a system S iff it is not the case that Γ is consistent in S, i.e. iff it is possible to derive a contradiction from Γ in S.

In the definition of a derivation we have assumed that the set Γ of premises in a tableau derivation of B from Γ is finite. The tableau technique can be extended to deal with (denumerably) infinite sets of premises by straightforwardly adapting a method mentioned by Smullyan [37], p. 64, but we will not labour the details here.

4.5. The logic of a tableau system

Let S be a tableau system. Then the logic (or the (logical) system) of S, $L(S)$, is the set of all sentences (in L) that are provable in S, in symbols $L(S) = \{A \in L\colon \vdash_S A\}$. E.g. $L(aTdDt4)$ is the set of all sentences that are provable in the system $aTdDt4$, i.e. in the system that includes the basic rules and the (non-basic) rules T-aT, T-dD and T-$t4$.

5. Examples of theorems

Theorem 1. *The sentences in Tables 15–27 are theorems (or more precisely theorem schemas) in the indicated systems.*

Proof. Straightforward. □

The pure temporal, alethic and deontic conditions on models and theorems discussed above are well-known in the literature, and require no further comment here. Let S be a tableau system. By adding different a-rules (Table 10) to T in obvious ways S can be made to include any of the 32 standard normal alethic systems (including e.g. M, B, $S4$ and $S5$) for historical necessity and by adding different d-rules (Table 12) to T in obvious ways S can be made to include any of the 32 standard normal deontic systems (e.g. OK, SDL (standard deontic logic) and $OS5+$) well-known in the literature

Table 15

Name	Theorem	System
aK	$\Box(A \to B) \to (\Box A \to \Box B)$	T
aT	$\Box A \to A$	aT
aD	$\Box A \to \Diamond A$	aD
aB	$A \to \Box\Diamond A$	aB
$a4$	$\Box A \to \Box\Box A$	$a4$
$a5$	$\Diamond A \to \Box\Diamond A$	$a5$

Table 16

Name	Theorem	System
dK	$O(A \to B) \to (OA \to OB)$	T
dD	$OA \to PA$	dD
$d4$	$OA \to OOA$	$d4$
$d5$	$PA \to OPA$	$d5$
dT'	$O(OA \to A)$	dT'
dB'	$O(POA \to A)$	dB'

Table 17

Name	Theorem	System
tFK	$\underline{G}(A \to B) \to (\underline{G}A \to \underline{G}B)$	T
tPK	$\underline{H}(A \to B) \to (\underline{H}A \to \underline{H}B)$	T
	$A \to \underline{G}\underline{P}A$	T
	$A \to \underline{H}\underline{F}A$	T
$t4$	$\underline{G}A \to \underline{G}\underline{G}A$	$t4$
$t4'$	$\underline{H}A \to \underline{H}\underline{H}A$	$t4$
FD	$\underline{G}A \to \underline{F}A$	tFD
PD	$\underline{H}A \to \underline{P}A$	tPD
DE	$\underline{F}A \to \underline{F}\underline{F}A$	tDE
DE'	$\underline{P}A \to \underline{P}\underline{P}A$	tDE
FC	$(\underline{F}A \wedge \underline{F}B) \to (\underline{F}(A \wedge \underline{F}B) \vee \underline{F}(A \wedge B) \vee \underline{F}(\underline{F}A \wedge B))$	tFC
PC	$(\underline{P}A \wedge \underline{P}B) \to (\underline{P}(A \wedge \underline{P}B) \vee \underline{P}(A \wedge B) \vee \underline{P}(\underline{P}A \wedge B))$	tPC

Table 18

Name	Theorem	System
MO	$\Box A \to OA$	$adMO$
OC	$OA \to \Diamond A$	$adOC$
OC'	$O(OA \to \Diamond A)$	$adOC'$
MO'	$O(\Box A \to OA)$	$adMO'$
$ad4$	$OA \to \Box OA$	$ad4$
$ad5$	$PA \to \Box PA$	$ad5$
PMP	$P\Box A \to \Box PA$	$adPMP$
OMP	$O\Box A \to \Box OA$	$adOMP$
MOP	$\Box OA \to O\Box A$	$adMOP$

Table 19
Theorems in T.

$\neg RtA \leftrightarrow Rt\neg A$	$Rt(A \vee B) \leftrightarrow (RtA \vee RtB)$
$RtA \leftrightarrow \neg Rt\neg A$	$Rt(A \to B) \leftrightarrow (RtA \to RtB)$
$Rt(A \wedge B) \leftrightarrow (RtA \wedge RtB)$	$Rt(A \leftrightarrow B) \leftrightarrow (RtA \leftrightarrow RtB)$

(see e.g. Chellas [16] and Åqvist [1]). Likewise, S can be made to include many standard temporal systems by adding t-rules (Table 11) to T in obvious ways.

Let the concepts of an $S5$-operator, M-operator, etc. be understood in the usual way. Then U and M are $S5$-operators in every system. In T $[G]$, $\langle F\rangle$, $[H]$ and $\langle P\rangle$ are M-operators and $\underline{A}$ and $\underline{S}$ are B-operators. In $t4$ $[G]$, $\langle F\rangle$, $[H]$ and $\langle P\rangle$ are $S4$-operators and $\underline{A}$ and $\underline{S}$ are B-operators. In $t4PC$ $[G]$, $\langle F\rangle$, $[H]$ and $\langle P\rangle$ are $S4$-operators. In $t4PCFC$ and $t4PCFCDE$ $[G]$,

Table 20

Name	Theorem	System
	$Rt'RtA \leftrightarrow RtA$	T
	$Rt\Box A \leftrightarrow Rt\Box RtA$	T
	$RtOA \leftrightarrow RtORtA$	T
	$Rt\Diamond A \leftrightarrow Rt\Diamond RtA$	T
	$RtPA \leftrightarrow RtPRtA$	T
	$\underline{F}\underline{G}(A \wedge \neg A) \rightarrow \Box \underline{F}\underline{G}(A \wedge \neg A)$	T
	$\underline{G}\underline{F}(A \vee \neg A) \rightarrow \Box \underline{G}\underline{F}(A \vee \neg A)$	T
	$(\underline{G}A \rightarrow \underline{F}A) \rightarrow \Box(\underline{G}A \rightarrow \underline{F}A)$	T
	$(\underline{H}A \rightarrow \underline{P}A) \rightarrow \Box(\underline{H}A \rightarrow \underline{P}A)$	T
	$\underline{P}\underline{H}(A \wedge \neg A) \rightarrow \Box \underline{P}\underline{H}(A \wedge \neg A)$	T
	$\underline{H}\underline{P}(A \vee \neg A) \rightarrow \Box \underline{H}\underline{P}(A \vee \neg A)$	T
	$RtA \rightarrow Rt\Box RtA$, where A is atomic	$adtFT$
SP	$Rt'\Box RtA \rightarrow Rt\Box RtA$, if $v(t') < v(t)$	$adtSP$
SR	$Rt'O(Rt'ORtA \rightarrow RtORtA)$, if $v(t') < v(t)$	$adtSR$
PI	$Rt'ORt(OA \rightarrow \Box A)$, if $v(t') < v(t)$	$adtPI$

Table 21
Theorems in $adtSP$.

$\underline{H}\Box A \rightarrow \Box \underline{H}A$	$\Diamond \underline{P}A \rightarrow \underline{P}\Diamond A$
$\underline{P}\Box A \rightarrow \Box \underline{P}A$	$\Diamond \underline{H}A \rightarrow \underline{H}\Diamond A$
$\Box \underline{G}A \rightarrow \underline{G}\Box A$	$\underline{F}\Diamond A \rightarrow \Diamond \underline{F}A$
$\underline{P}\diamondminus A \rightarrow \diamondminus \underline{H}A$	$\Box \underline{G}\neg A \rightarrow \underline{G}\diamondminus A$
$\underline{H}\diamondminus A \rightarrow \Box \underline{H}\neg A$	$\Box \underline{G}(A \rightarrow B) \rightarrow \underline{G}(\Box A \rightarrow \Box B)$
$\underline{H}\diamondminus A \rightarrow \diamondminus \underline{P}A$	$\Box \underline{G}(A \leftrightarrow B) \rightarrow \underline{G}(\Box A \leftrightarrow \Box B)$
$\underline{H}\Box(A \rightarrow B) \rightarrow \Box(\underline{H}A \rightarrow \underline{H}B)$	$\Box(\underline{G}A \wedge \underline{G}B) \rightarrow \underline{G}(\Box A \wedge \Box B)$
$\underline{H}(A \Rightarrow B) \rightarrow (\underline{H}A \Rightarrow \underline{H}B)$	$\underline{F}(\Diamond A \vee \Diamond B) \rightarrow \Diamond(\underline{F}A \vee \underline{F}B)$
$\underline{H}\Box(A \leftrightarrow B) \rightarrow \Box(\underline{H}A \leftrightarrow \underline{H}B)$	$\Box(\underline{G}\neg A \wedge \underline{G}\neg B) \rightarrow \underline{G}(\diamondminus A \wedge \diamondminus B)$
$\underline{H}(A \Leftrightarrow B) \rightarrow (\underline{H}A \Leftrightarrow \underline{H}B)$	$(\Box \underline{G}A \vee \Box \underline{G}B) \rightarrow \underline{G}(\Box A \vee \Box B)$
$\underline{H}(\Box A \wedge \Box B) \rightarrow \Box(\underline{H}A \wedge \underline{H}B)$	$\underline{F}(\Diamond A \wedge \Diamond B) \rightarrow (\Diamond \underline{F}A \wedge \Diamond \underline{F}B)$
$\Diamond(\underline{P}A \vee \underline{P}B) \rightarrow \underline{P}(\Diamond A \vee \Diamond B)$	$(\Box \underline{G}\neg A \vee \Box \underline{G}\neg B) \rightarrow \underline{G}(\diamondminus A \vee \diamondminus B)$
$\underline{H}(\diamondminus A \wedge \diamondminus B) \rightarrow \Box(\underline{H}\neg A \wedge \underline{H}\neg B)$	$\underline{P}(\Box A \wedge \Box B) \rightarrow \Box(\underline{P}A \wedge \underline{P}B)$
$(\underline{H}\Box A \vee \underline{H}\Box B) \rightarrow \Box(\underline{H}A \vee \underline{H}B)$	$\Diamond(\underline{H}A \wedge \underline{H}B) \rightarrow \underline{H}(\Diamond A \wedge \Diamond B)$
$\Diamond(\underline{P}A \wedge \underline{P}B) \rightarrow (\underline{P}\Diamond A \wedge \underline{P}\Diamond B)$	$\Diamond(\underline{H}A \vee \underline{H}B) \rightarrow \underline{H}(\Diamond A \vee \Diamond B)$
$(\underline{H}\diamondminus A \vee \underline{H}\diamondminus B) \rightarrow \Box(\underline{H}\neg A \vee \underline{H}\neg B)$	$\underline{P}(\Box A \vee \Box B) \rightarrow \Box(\underline{P}A \vee \underline{P}B)$

Table 22
Theorems in $aTdDadMO$.

$(\Box A \vee \Box \neg A) \rightarrow (OA \leftrightarrow A)$	$\triangle\, A \rightarrow (OA \leftrightarrow A)$
$(\Box A \vee \Box \neg A) \rightarrow (PA \leftrightarrow A)$	$\triangle\, A \rightarrow (PA \leftrightarrow A)$
$(\Box A \vee \Box \neg A) \rightarrow (\Box A \leftrightarrow A)$	$\triangle\, A \rightarrow (\Box A \leftrightarrow A)$
$(\Box A \vee \Box \neg A) \rightarrow (\Diamond A \leftrightarrow A)$	$\triangle\, A \rightarrow (\Diamond A \leftrightarrow A)$

$\langle F\rangle$, $[H]$ and $\langle P\rangle$ are $S4.3$-operators and $\underline{A}$ and $\underline{S}$ are $S5$-operators. Obviously, $[G]A \rightarrow \underline{G}A$, $[H]A \rightarrow \underline{H}A$, $\underline{F}A \rightarrow \langle F\rangle A$ and $\underline{P}A \rightarrow \langle P\rangle A$ are theorem schemas in every system.

Theorem 2.

(i) *$\underline{A}A \rightarrow \underline{G}\underline{H}A$ and $\underline{A}A \rightarrow [G][H]A$ are theorem schemas in tPC.*
(ii) *$\underline{A}A \rightarrow \underline{H}\underline{G}A$ and $\underline{A}A \rightarrow [H][G]A$ are theorem schemas in tFC.*
(iii) *Let $[*] = \Box$, O, $\underline{A}$, Rt, $[G]$, $\underline{G}$, $[H]$ or $\underline{H}$, and $\langle * \rangle = \Diamond$, P, $\underline{S}$, Rt, $\langle F\rangle$, $\underline{F}$, $\langle P\rangle$ or $\underline{P}$. Then $UA \rightarrow [*]A$ and $\langle * \rangle A \rightarrow MA$ are theorem schemas in T.*
(iv) *$UA \rightarrow \Box \underline{A}A$, $UA \rightarrow \underline{A}\Box A$, $\Diamond \underline{S}A \rightarrow MA$ and $\underline{S}\Diamond A \rightarrow MA$ are theorem schemas in T.*

Proof. Straightforward. $\Box$

Table 23
Theorems in $aTdDadMO$.

$\triangle A \to (O(A \wedge B) \leftrightarrow (A \wedge OB))$
$\triangle A \to (O(B \wedge A) \leftrightarrow (OB \wedge A))$
$\triangle A \to (O(A \vee B) \leftrightarrow (A \vee OB))$
$\triangle A \to (O(B \vee A) \leftrightarrow (OB \vee A))$
$\triangle A \to (O(A \to B) \leftrightarrow (A \to OB))$
$\triangle A \to (O(B \to A) \leftrightarrow (PB \to A))$
$\triangle A \to (O(A \leftrightarrow B) \leftrightarrow ((A \to OB) \wedge (PB \to A)))$
$\triangle A \to (O(B \leftrightarrow A) \leftrightarrow ((PB \to A) \wedge (A \to OB)))$

The bi-modal, alethic–deontic conditions, rules and theorems mentioned above are discussed more closely in *Bimodal Logic* and *Bimodal Logic and Semantic Tableaux* (see Section 3.2). In these essays several sentences that become provable if we add some of the ad-rules are listed, and we shall only add a few comments about them here. (Kracht [30] and Gabbay, Kurucz, Wolter and Zakharyaschev [26] contain some more general ideas about how to combine two or more modal systems.) $OA \to \Diamond A$ and $O(OA \to \Diamond A)$ are two versions of Kant's ought-implies-can principle that are provable in some of our systems. The sentence $(OA \wedge \Box(A \to B)) \to OB$ is a theorem in any system including T-MO. This sentence is a version of the so-called means-end principle. $OA \to \Box OA$ says that all obligations are historically necessary, $PA \to \Box PA$ that all permissions are historically necessary and $FA \to \Box FA$ that all prohibitions are historically necessary. $OA \to \Box OA$ and $FA \to \Box FA$ are theorem schemas in all systems that include T-$ad4$ and $PA \to \Box PA$ is a theorem schema in all systems that include T-$ad5$. So, if we add T-$ad4$ and T-$ad5$ to T all norms become historically necessary. However, $OA \to UOA$, $PA \to UPA$ and $FA \to UFA$ are not theorems. Our norms are therefore not absolute.

With these brief remarks out of the way, let us concentrate on the conditions, rules and theorems that are characteristic of our systems in this essay.

We shall say that A is non-future iff A doesn't contain any operator of the form $\underline{G}$, $\underline{F}$, or Rt (at least if $v(t)$ is a time later than the time of the valuation of the sentence).

Theorem 3.

(i) $p \to \Box p$, $\neg p \to \Box\neg p$, $\Box p \vee \Box\neg p$, $\triangle p$ *and* $(pCq) \to \Box(pCq)$, *where* $C = \wedge, \vee, \to$ *or* $\leftrightarrow$, *are theorems in* $adtFTBT$.
(ii) $UA \to \Box UA$ *and* $MA \to \Box MA$ *hold in* T.
(iii) $\underline{H}p \to \Box\underline{H}p$ *and* $\underline{P}p \to \Box\underline{P}p$ *are theorems in* $adtSPFT$.

Assume that $v(t') < v(t)$ *in* (iv)–(x). *Then*

(iv) $Rt'p \to Rt\Box Rt'p$, $Rt'\neg p \to Rt\Box Rt'\neg p$ *and* $Rt'(pCq) \to Rt\Box Rt'(pCq)$, *where* $C = \wedge, \vee, \to$ *or* $\leftrightarrow$, *are theorems in* $adtFTBTSP$.
(v) $Rt'\underline{H}p \to Rt\Box Rt'\underline{H}p$ *and* $Rt'\underline{P}p \to Rt\Box Rt'\underline{P}p$ *are theorems in* $t4adtSPFT$.
(vi) $Rt'\Box A \to Rt\Box Rt'\Box A$ *is a theorem schema in* $a4adtSP$.
(vii) $Rt'\Diamond A \to Rt\Box Rt'\Diamond A$ *is provable in* $a5adtSP$.
(viii) $Rt'OA \to Rt\Box Rt'OA$ *hold in* $ad4adtSP$.
(ix) $Rt'PA \to Rt\Box Rt'PA$ *is a theorem schema in* $ad5adtSP$.
(x) $Rt'UA \to Rt\Box Rt'UA$ *and* $Rt'MA \to Rt\Box Rt'MA$ *are provable in* T.
(xi) *If* A *is non-future, then* $\Box A$ *and* $\triangle A$ *are theorem schemas in the systems* $aTB4ad45adtSPFT$ *and* $aTB4ad45adtSPBT$.
(xii) *If* A *is non-future, then all of the following sentences are theorems in the systems* $aTB4dDad45MOadtSPFT$ *and* $aTB4dDad45MOadtSPBT$: $\Box A \leftrightarrow A$, $OA \leftrightarrow A$, $\Diamond A \leftrightarrow A$, $PA \leftrightarrow A$.
(xiii) *If* A *is non-future and* $v(t') < v(t)$, *then* $Rt'A \to Rt\Box Rt'A$.

Proof. Straightforward. (Note that T-BT is a derived rule in every system that includes T-FT and T-aB and that T-FT is a derived rule in every system that includes T-BT and T-aB.) $\Box$

So, there are systems in which the distinctions between some different modalities collapse under certain circumstances, i.e. all of the following sentences are equivalent: $\Box A$, OA, A, PA, $\Diamond A$ in the systems indicated in Theorem 3, part (xii), given that A is non-future. This suggests that genuine norms concern the future and that O and P only function as practical deontic operators when they are prefixed to future-tensed sentences.

Theorem 4. *If* A *is non-future, then all of the consequents in* Table 23 (*i.e.* $O(A \wedge B) \leftrightarrow (A \wedge OB)$, $O(B \wedge A) \leftrightarrow (OB \wedge A)$, *etc.*) *are theorems in the systems* $aTB4dDad45MOadtSPFT$ *and* $aTB4dDad45MOadtSPBT$.

Proof. Straightforward. $\Box$

Table 24
Theorems in $aTdDadMO$.

$\triangle A \to (P(A \wedge B) \leftrightarrow (A \wedge PB))$
$\triangle A \to (P(B \wedge A) \leftrightarrow (PB \wedge A))$
$\triangle A \to (P(A \vee B) \leftrightarrow (A \vee PB))$
$\triangle A \to (P(B \vee A) \leftrightarrow (PB \vee A))$
$\triangle A \to (P(B \to A) \leftrightarrow (OB \to A))$
$\triangle A \to (P(A \to B) \leftrightarrow (A \to PB))$
$\triangle A \to (P(A \leftrightarrow B) \leftrightarrow ((A \to PB) \wedge (OB \to A)))$
$\triangle A \to (P(B \leftrightarrow A) \leftrightarrow ((OB \to A) \wedge (A \to PB)))$

Table 25
Theorems in tPD.

$U\underline{G}A \to \underline{G}UA$	$\underline{F}MA \to M\underline{F}A$
$\underline{H}UA \to U\underline{H}A$	$M\underline{P}A \to \underline{P}MA$
$\underline{P}UA \to U\underline{P}A$	$M\underline{H}A \to \underline{H}MA$

Table 26
Theorems in tFD.

$\underline{G}UA \to U\underline{G}A$	$M\underline{F}A \to \underline{F}MA$
$U\underline{H}A \to \underline{H}UA$	$\underline{P}MA \to M\underline{P}A$
$\underline{F}UA \to U\underline{F}A$	$M\underline{G}A \to \underline{G}MA$

Table 27
Theorems in some systems.

Theorem	System	Theorem	System
$U\Box A \to \Box UA$	aT	$\Diamond MA \to M\Diamond A$	aT
$\Box UA \to U\Box A$	aD and aT	$M\Diamond A \to \Diamond MA$	aD and aT
$\Diamond UA \to U\Diamond A$	aD and aT	$M\Box A \to \Box MA$	aD and aT

Some philosophers think that a conditional obligation should be formalized as a "wide conditional obligation" of the form $O(A \to B)$ and others think that it should be formalized as a "narrow conditional obligation" of the form $A \to OB$. (Of course, many philosophers reject both of these views and argue for some alternative. However, the classical alternatives are captured by these symbolizations.) As we can see, these formalizations are equivalent in some systems under certain conditions. $O(A \to B)$ is equivalent to $A \to OB$, when A is historically non-contingent, in the system $aTdDadMO$, and in the systems mentioned in Theorem 4 these sentences are equivalent when A is non-future. Factual detachment is not a valid inference pattern for wide conditional obligations, i.e. we cannot always derive OB from A and $O(A \to B)$. However, in the system $aTdDadMO$ we can detach OB from A and $O(A \to B)$ when A is historically settled or non-contingent, and in the systems $aTB4dDad45MOadtSPFT$ and $aTB4dDad45MOadtSPBT$ we can detach OB from the same sentences when A is non-future. (These facts may shed some light on the so-called dilemma of commitment and detachment, see van Eck [21], Chapters II and IV.)

Theorem 5. *If A is non-future, then all of the consequents in Table 24 (i.e. $P(A \wedge B) \leftrightarrow (A \wedge PB)$, $P(B \wedge A) \leftrightarrow (PB \wedge A)$, etc.) are theorems in the systems $aTB4dDad45MOadtSPFT$ and $aTB4dDad45MOadtSPBT$.*

Proof. Straightforward. $\Box$

Bailhache has suggested that we should impose condition C-PI on a reasonable temporal alethic–deontic logic. If we assume this condition, $Rt'ORt(OA \to \Box A)$, if $v(t') < v(t)$ is valid (see e.g. Bailhache [5] or [6]). This sentence can be proved in all our systems that include T-PI. The reason that Bailhache wants to impose this condition is that he thinks that a deontically accessible world cannot lead in the future to a world that is not deontically accessible. Nevertheless, intuitively PI may seem to be too strong. We have suggested a slightly weaker condition C-WPI that seems to be a reasonable condition given certain interpretations, for instance if deontic accessibility is defined in the following way and we want the preference relation between possible worlds to be the same at every time: $\forall t \forall x \forall y(Sxyt \leftrightarrow (Rxyt \wedge \neg\exists z(Rxzt \wedge Pzy)))$, where t ranges over times in T, x and y over possible worlds in W, S and R are interpreted as usual, and P is some kind of strong binary preference relation between possible worlds in W. (We have not been able to find any sentence that would define this condition in an axiomatic formulation of our systems.) C-WPI follows from C-PI, but not vice versa. Someone who thinks that C-PI is too strong may nevertheless want our models to satisfy C-WPI and add T-WPI to our tableau systems. If we define the deontic accessibility relation as suggested above the condition $C\text{-}SR$ is also a reasonable condition. (For more information on the Rt operator, see Rescher and Urquhart [36].)

Thomason [40] mentions two sentences that he takes to be reasonable (and that are valid on some classes of models that he discusses): $O\underline{G}(\underline{F}p \to \neg O\underline{G}\neg p)$ and $O\underline{G}p \to O\underline{G}Op$. These sentences can be proved in some of our systems.

Theorem 6.

(i) *$O\underline{G}A \to O\underline{G}OA$ is a theorem (schema) in the system $d4adtSR$.*
(ii) *$O\underline{G}(\underline{F}A \to \neg O\underline{G}\neg A)$ and $O\underline{G}(\underline{F}A \to P\underline{F}A)$ are theorem (schemas) in the systems $aTadtPI$ and $aTdT'adtWPI$.*

Proof. Straightforward. □

This suffices to illustrate some of the characterizing features of our temporal alethic–deontic logics.

Example 7. Let us return to the argument mentioned in the introduction, the *You ought to love your children* argument. We will now show that this is valid in the class of all models if the concept of necessity is interpreted as absolute necessity. Let l stand for "You love your children", r for "You respect your children" and c for "You care for your children". Then the argument is symbolized in the following way: $\underline{G}Ol$, $U(l \to (r \wedge c)) \therefore \underline{G}Or \wedge \underline{G}Oc$. The T-tableau for this is closed (see below). Hence, the conclusion is derivable from the premises in T. It follows that this argument is valid in the class of all models. For T is sound with respect to this class.

$$
\begin{array}{c}
(1)\ \underline{G}Ol,\ w0t0 \\
(2)\ U(l \to (r \wedge c)),\ w0t0 \\
(3)\ \neg(\underline{G}Or \wedge \underline{G}Oc),\ w0t0 \\
\swarrow \qquad \searrow \\
\begin{array}{r|l}
(4)\ \neg\underline{G}Or,\ w0t0\ [3, \neg\wedge] & (5)\ \neg\underline{G}Oc,\ w0t0\ [3, \neg\wedge] \\
(6)\ \underline{F}\neg Or,\ w0t0\ [4, \neg\underline{G}] & (7)\ \underline{F}\neg Oc,\ w0t0\ [5, \neg\underline{G}] \\
(8)\ t0 < t1\ [6, \underline{F}] & (9)\ t0 < t1\ [7, \underline{F}] \\
(10)\ \neg Or,\ w0t1\ [6, \underline{F}] & (11)\ \neg Oc,\ w0t1\ [7, \underline{F}] \\
(12)\ P\neg r,\ w0t1\ [10, \neg O] & (13)\ P\neg c,\ w0t1\ [11, \neg O] \\
(14)\ sw0w1t1\ [12, P] & (15)\ sw0w1t1\ [13, P] \\
(16)\ \neg r,\ w1t1\ [12, P] & (17)\ \neg c,\ w1t1\ [13, P] \\
(18)\ Ol,\ w0t1\ [1, 8, \underline{G}] & (19)\ Ol,\ w0t1\ [1, 9, \underline{G}] \\
(20)\ l,\ w1t1\ [18, 14, O] & (21)\ l,\ w1t1\ [19, 15, O] \\
(22)\ l \to (r \wedge c),\ w1t1\ [2, U] & (23)\ l \to (r \wedge c),\ w1t1\ [2, U] \\
\swarrow \qquad \searrow & \swarrow \qquad \searrow \\
\begin{array}{l|l}
(24)\ \neg l,\ w1t1 & (25)\ r \wedge c,\ w1t1\ [22, \to] \\
(28)\ *\ [20, 24] & (29)\ r,\ w1t1\ [25, \wedge] \\
 & (32)\ c,\ w1t1\ [25, \wedge] \\
 & (34)\ *\ [16, 29]
\end{array} &
\begin{array}{l|l}
(26)\ \neg l,\ w1t1 & (27)\ r \wedge c,\ w1t1\ [23, \to] \\
(30)\ *\ [21, 26] & (31)\ r,\ w1t1\ [27, \wedge] \\
 & (33)\ c,\ w1t1\ [27, \wedge] \\
 & (35)\ *\ [17, 33]
\end{array}
\end{array}
\end{array}
$$

Example 8. Suppose that we interpret the concept of necessity in the second premise in the *You ought to love your children* argument as historical necessity. Then it is symbolized in the following way: $\underline{G}Ol$, $\Box(l \to (r \wedge c)) \therefore \underline{G}Or \wedge \underline{G}Oc$, where the atomic sentences are interpreted as in Example 7. This argument is invalid in the class of all models. We will now show this. The T-tableau for this argument (see below) is open and complete, i.e. it is not possible to apply any more T-rules. Hence, the conclusion is not derivable from the premises in the tableau system T. Since T is complete with respect to the class of all models, it follows that the argument is invalid in the class of all models. We can read off the following countermodel from the (leftmost) open branch: $W = \{\omega_0, \omega_1\}$, $T = \{\tau_0, \tau_1\}$, $\tau_0 < \tau_1$, R is empty, $S\omega_0\omega_1\tau_1$, l is true in ω_1 at τ_1, and r is false in ω_1 at τ_1. It is easy to see that the premises are true (in ω_0 at τ_0) and the conclusion false (in ω_0 at τ_0) in this model.

$$
\begin{array}{c}
(1)\ \underline{G}Ol,\ w0t0 \\
(2)\ \Box(l \to (r \wedge c)),\ w0t0 \\
(3)\ \neg(\underline{G}Or \wedge \underline{G}Oc),\ w0t0 \\
\swarrow \qquad \searrow \\
\begin{array}{r|l}
(4)\ \neg\underline{G}Or,\ w0t0\ [3, \neg\wedge] & (5)\ \neg\underline{G}Oc,\ w0t0\ [3, \neg\wedge] \\
(6)\ \underline{F}\neg Or,\ w0t0\ [4, \neg\underline{G}] & (7)\ \underline{F}\neg Oc,\ w0t0\ [5, \neg\underline{G}] \\
(8)\ t0 < t1\ [6, \underline{F}] & (9)\ t0 < t1\ [7, \underline{F}] \\
(10)\ \neg Or,\ w0t1\ [6, \underline{F}] & (11)\ \neg Oc,\ w0t1\ [7, \underline{F}] \\
(12)\ P\neg r,\ w0t1\ [10, \neg O] & (13)\ P\neg c,\ w0t1\ [11, \neg O]
\end{array}
\end{array}
$$

(14) $sw0w1t1$ [12, P]	(15) $sw0w1t1$ [13, P]
(16) $\neg r, w1t1$ [12, P]	(17) $\neg c, w1t1$ [13, P]
(18) $Ol, w0t1$ [1, 8, $\underline{G}$]	(19) $Ol, w0t1$ [1, 9, $\underline{G}$]
(20) $l, w1t1$ [18, 14, O]	(21) $l, w1t1$ [19, 15, O]

6. Soundness and completeness theorems

We are now in a position to prove that all the systems in this essay are sound and complete with respect to their semantics. To the best of our knowledge these proofs are completely new, even though the basic idea behind the proofs is the same as the idea behind similar soundness and completeness proofs for propositional logic and modal logic where a sentence is evaluated at one parameter (world, moment in time, point) (see e.g. [29,33,37]).

Let $S = aA_1 \ldots A_i dB_1 \ldots B_j adC_1 \ldots C_k tD_1 \ldots D_l adtE_1 \ldots E_m$ be a temporal alethic–deontic tableau system as defined above. Then we shall say that the class of models, $\mathbf{M}$, corresponds to S just in case $\mathbf{M} = \mathbf{M}(C-A_1, \ldots, C-A_i, C-B_1, \ldots, C-B_j, C-C_1, \ldots, C-C_k, C-D_1, \ldots, C-D_l, C-E_1, \ldots, C-E_m)$.

S is strongly sound with respect to $\mathbf{M}$ iff $\Gamma \vdash_S A$ entails $\mathbf{M}, \Gamma \Vdash A$. S is strongly complete with respect to $\mathbf{M}$ just in case $\mathbf{M}, \Gamma \Vdash A$ entails $\Gamma \vdash_S A$.

6.1. Soundness theorems

Let M be any temporal alethic–deontic model and b any branch of a tableau. Then b is satisfiable in M iff there is a function f from $w0, w1, w2, \ldots$ to W and a function g from $t0, t1, t2, \ldots$ to T such that (i) A is true in $f(wi)$ at $g(tj)$ in M, for every node $A, witj$ on b, (ii) if $rwiwjtk$ is on b, then $Rf(wi)f(wj)g(tk)$ in M, (iii) if $swiwjtk$ is on b, then $Sf(wi)f(wj)g(tk)$ in M, (iv) if $ti < tj$ is on b, then $g(ti) < g(tj)$ in M, (v) if $ti = tj$ is on b, then $g(ti) = g(tj)$ in M. If these conditions are fulfilled, we say that f and g show that b is satisfiable in M.

Lemma 9 *(Soundness lemma). Let b be any branch of a tableau and M be any (temporal alethic–deontic) model. If b is satisfiable in M and a tableau rule is applied to it, then it produces at least one extension, b', of b such that b' is satisfiable in M.*

Proof. The proof proceeds by going through all the tableau rules. We only sketch some parts to illustrate the method.

($\wedge$) Suppose that $A \wedge B, witj$ is on b, and that we apply ($\wedge$) to give an extended branch, b', of b containing $A, witj$ and $B, witj$. Since b is satisfiable in M there is a function f and a function g such that $A \wedge B$ is true in $f(wi)$ at $g(tj)$. Hence, A is true in $f(wi)$ at $g(tj)$ and B is true in $f(wi)$ at $g(tj)$.

($\neg\wedge$) Suppose that $\neg(A \wedge B), witj$ is on b, and that we apply ($\neg\wedge$) to it. Then two branches arise, one extending b with $\neg A, witj$ (the left branch); the other extending b with $\neg B, witj$ (the right branch). Since b is satisfiable in M there is a function f and a function g such that $\neg(A \wedge B)$ is true in $f(wi)$ at $g(tj)$. Hence, $\neg A$ is true in $f(wi)$ at $g(tj)$ or $\neg B$ is true in $f(wi)$ at $g(tj)$. In the first case, the left branch is satisfiable in M. In the second case, the right branch is satisfiable in M.

($\Box$) Suppose that $\Box A, witj$ and $rwiwjtk$ are on b and that we apply ($\Box$) to give an extended branch, b', of b containing $A, wjtk$. Since b is satisfiable in M there is a function f and a function g such that $\Box A$ is true in $f(wi)$ at $g(tj)$ in M and $Rf(wi)f(wj)g(tk)$. Hence, A is true in $f(wj)$ at $g(tk)$.

($\neg\Box$) Assume that $\neg\Box A, witj$ is on b and that we apply ($\neg\Box$) to obtain an extension of b, b', containing $\Diamond\neg A, witj$. Now b is satisfiable in M. So, there is a function f and a function g such that $\neg\Box A$ is true in $f(wi)$ at $g(tj)$ in M. Hence, $\Diamond\neg A$ is true in $f(wi)$ at $g(tj)$ in M.

($\neg P$) Suppose that $\neg PA, witj$ is on b, and that we apply the rule ($\neg P$) to extend the branch with $O\neg A, witj$. Since b is satisfiable in M there is a function f and a function g such that $\neg PA$ is true in $f(wi)$ at $g(tj)$. Consequently, $O\neg A$ is true in $f(wi)$ at $g(tj)$.

(P) Assume that $PA, witk$ is on b and that we apply (P) to obtain an extension, b', of b with two new nodes of the form $swiwjtk$ and $A, wjtk$ on b. Since b is satisfiable in M there is a function f and a function g such that PA is true in $f(wi)$ at $g(tk)$. Hence, for some $\omega \in W$, $Sf(wi)\omega g(tk)$ and A is true in ω at $g(tk)$. Let f' be the same as f except that $f'(wj) = \omega$. Note that f' and g also show that b is satisfiable in M, since f and f' differ only at wj. Moreover, by definition, $Sf'(wi)f'(wj)g(tk)$, and A is true in $f'(wj)$ at $g(tk)$. So, f' and g show that b' is satisfiable in M.

(T-$d4$) Suppose that $swiwjtl$ and $swjwktl$ are on b, and that we apply (T-$d4$) to give an extended branch b' of b containing $swiwktl$. Now there is a function f and a function g such that $Sf(wi)f(wj)g(tl)$ and $Sf(wj)f(wk)g(tl)$. For b is satisfiable in M. Accordingly, $Sf(wi)f(wk)g(tl)$. For M satisfies (C-$d4$).

(T-aB) Suppose that $rwiwjtk$ is on b, and that we apply (T-aB) to give an extended branch b' of b containing $rwjwitk$. Since b is satisfiable in M there is a function f and a function g such that $Rf(wi)f(wj)g(tk)$. Hence, $Rf(wj)f(wi)g(tk)$. For M satisfies (C-aB).

(T-DE) Suppose that $ti < tj$ is on b, and that we apply (T-DE) to get $ti < tk$ and $tk < tj$, where tk is new to the branch. Since b is satisfiable in M there is a function f and a function g such that $g(ti) < g(tj)$. Hence, for some τ in T, $g(ti) < \tau$ and $\tau < g(tj)$. For M satisfies condition (C-DE). Let g' be the same as g except that $g'(tk) = \tau$. Hence, $g'(ti) < g'(tk)$ and $g'(tk) < g'(tj)$. Since tk does not occur on the branch b, f and g' show that b' is satisfiable in M.

(T-FC) Suppose $ti < tj$ and $ti < tk$ are on b and that we apply (T-FC) to b. Then we get three branches, one extending b with $tj < tk$, one with $tj = tk$, and one with $tk < tj$. Since b is satisfiable in M there is a function f and a function g such that $g(ti) < g(tj)$ and $g(ti) < g(tk)$. By condition (C-FC), $g(tj) < g(tk)$ or $g(tk) < g(tj)$ or $g(tj) = g(tk)$. So, f and g show that at least one branch is satisfiable in M.

($Id(I)$) Assume that $A(ti)$ and $ti = tj$ are on b, and that we obtain $A(tj)$ by ($Id(I)$). Since b is satisfiable in M there is a function f and a function g such that $g(ti) = g(tj)$. If $A(ti)$ is $A, wkti$, A is true in $f(wk)$ at $g(ti)$. So, A is true in $f(wk)$ at $g(tj)$, which is what we wanted to show. If $A(ti)$ is $tk < ti$, $g(tk) < g(ti)$. So, $g(tk) < g(tj)$, as required. If $A(ti)$ is $ti = tk$, $g(ti) = g(tk)$. Hence, $g(tj) = g(tk)$, which is the result we wanted. If $A(ti)$ is $rwkwlti$, $Rf(wk)f(wl)g(ti)$. Consequently, $Rf(wk)f(wl)g(tj)$, as required.

(T-OC) Assume that $witk$ is on b, and that we apply (T-OC) to give an extended branch, b', of b containing $swiwjtk$ and $rwiwjtk$, where wj is new. Since b is satisfiable in M there is a function f and a function g such that $f(wi)$ is in W and $g(tk)$ in T. Hence, for some ω in W, $Sf(wi)\omega g(tk)$ and $Rf(wi)\omega g(tk)$, since condition (C-OC) obtains. Let f' be the same as f except that $f'(wj) = \omega$. Since wj does not occur on b, f' and g show that b is satisfiable in M. Moreover, $Sf'(wi)f'(wj)g(tk)$ and $Rf'(wi)f'(wj)g(tk)$ by construction. Hence, f' and g show that b' is satisfiable in M.

(T-FT) Suppose that $A, witk$ (where A is atomic) and $rwiwjtk$ are on b and that we apply (T-FT) to b to obtain an extension b' containing $A, wjtk$. Since b is satisfiable in M there is a function f and a function g such that A is true in $f(wi)$ at $g(tk)$ and $Rf(wi)f(wj)g(tk)$. Hence, A is true in $f(wj)$ at $g(tk)$. For M satisfies (C-FT) and A is atomic.

(T-SP) Suppose $rwiwjtl$ and $tk < tl$ are on b and that we apply (T-SP) to obtain an extension b' of b containing $rwiwjtk$. Since b is satisfiable in M there is a function f and a function g such that $Rf(wi)f(wj)g(tl)$ and $g(tk) < g(tl)$. Hence, $Rf(wi)f(wj)g(tk)$. For M satisfies (C-SP).

(T-SR) Suppose $swiwjtl$, $tl < tm$ and $swjwktm$ and that we apply (T-SR) to obtain an extension b' of b containing $swjwktl$. Since b is satisfiable in M there is a function f and a function g such that $Sf(wi)f(wj)g(tl)$, $g(tl) < g(tm)$ and $Sf(wj)f(wk)g(tm)$. Hence, $Sf(wj)f(wk)g(tl)$. For M satisfies condition (C-SR). □

Theorem 10 *(Soundness theorem). Let S be any of the tableau systems discussed in this essay and let* **M** *be the class of models that corresponds to S. Then S is strongly sound with respect to* **M**.

Proof. Once the Soundness lemma is established the proof is an easy modification of similar proofs that certain normal mono-modal systems are sound (see e.g. Priest [33]). □

6.2. Completeness theorems

Let b be an open complete branch of a tableau and let I be the set of numbers on b immediately preceded by a "t". We shall say that $i \rightleftharpoons j$ just in case $i = j$, or "$ti = tj$" or "$tj = ti$" occur on b. $\rightleftharpoons$ is an equivalence relation and $[i]$ is the equivalence class of i. The temporal alethic–deontic model, $M = \langle W, T, <, R, S, V, v\rangle$, induced by b is defined as follows. $W = \{\omega i\colon\ wi \text{ occurs on } b\}$, $T = \{\tau[i]\colon\ i \in I\}$, $\tau[i] < \tau[j]$ iff $ti < tj$ occurs on b, $R\omega i\omega j\tau[k]$ iff $rwiwjtk$ occurs on b, $S\omega i\omega j\tau[k]$ iff $swiwjtk$ occurs on b. If $p, witj$ occurs on b, then p is true in ωi at $\tau[j]$ (i.e. then $\langle \omega i, \tau[j]\rangle \in V(p)$); if $\neg p, witj$ occurs on b, then p is false in ωi at $\tau[j]$ (i.e. then it is not the case that $\langle \omega i, \tau[j]\rangle \in V(p)$). If ti occurs on b, then $v(ti) = \tau[i]$. We shall call this definition of an induced model "Def. IM". If our tableau system neither includes T-FC nor T-PC, $\rightleftharpoons$ is reduced to identity and $[i] = i$. Hence, in such systems, we may take T to be $\{\tau i\colon\ ti \textit{ occurs on } b\}$ and dispense with the equivalence classes.

Lemma 11 *(Completeness lemma). Let b be an open branch in a complete tableau and let M be a temporal alethic–deontic model induced by b. Then:*

(i) *A is true in ωi at $\tau[j]$, if $A, witj$ is on b, and*
(ii) *A is false in ωi at $\tau[j]$, if $\neg A, witj$ is on b.*

Proof. The proof is by induction on the complexity of A.

Basis. If A is atomic, the result is true by definition.

Induction step. We only go through some of the cases to illustrate the method.

$A = \neg B$. Suppose that $A, witj$ occurs on b, i.e. $\neg B, witj$ is on b. By the induction hypothesis, clause (ii) of the desired result holds for B, so B is false in ωi at $\tau[j]$, i.e. $\neg B$ is true in ωi at $\tau[j]$. Hence, if $\neg B, witj$ is on b, then $\neg B$ is true in ωi at $\tau[j]$; which result establishes clause (i) of the lemma. As for clause (ii), suppose that $\neg A, witj$ occurs on b, i.e. $\neg\neg B, witj$ is on b. Since the tableau is complete ($\neg\neg$) has been applied. Thus $B, witj$ is on b. Since, by the induction hypothesis, clause (i) of the desired result holds for B, B is true in ωi at $\tau[j]$, i.e. $\neg B$ is false in ωi at $\tau[j]$. Hence, if $\neg\neg B, witj$ is on b, then $\neg B$ is false in ωi at $\tau[j]$, which result settles clause (ii) of the lemma.

$A = B \vee C$. Suppose that $A, witj$ occurs on b, i.e. $B \vee C, witj$ is on b. Since the tableau is complete ($\vee$) has been applied. Thus either $B, witj$ or $C, witj$ is on b. By the induction hypothesis, either B is true in ωi at $\tau[j]$ or C is true in ωi at $\tau[j]$. Hence, $B \vee C$ is true in ωi at $\tau[j]$. Suppose that $\neg A, witj$ occurs on b, i.e. $\neg(B \vee C), witj$ is on b. Since the tableau is complete $(\neg\vee)$ has been applied. Thus both $\neg B, witj$ and $\neg C, witj$ are on b. By the induction hypothesis, B is false in ωi at $\tau[j]$ and C is false in ωi at $\tau[j]$. Hence $B \vee C$ is false in ωi at $\tau[j]$.

$A = \Box B$. Suppose A occurs on b, i.e. $\Box B, witk$ is on b. Since b is complete $(\Box)$ has been applied to $\Box B, witk$. Thus, for all wj such that $rwiwjtk$ is on b, $B, wjtk$ is on b. By the induction hypothesis, for all ωj such that $R\omega i\omega j\tau[k]$, B is true in ωj at $\tau[k]$. Hence, $\Box B$ is true in ωi at $\tau[k]$. Suppose that $\neg A$ occurs on b, i.e. $\neg\Box B, witk$ is on b. Then $\Diamond\neg B, witk$ is on b (by $(\neg\Box)$). For b is complete. Furthermore, since b is complete $(\Diamond)$ has been applied to $\Diamond\neg B, witk$. Thus, for some wj, $rwiwjtk$ and $\neg B, wjtk$ are on b. By the induction hypothesis, $R\omega i\omega j\tau[k]$ and B is false in ωj at $\tau[k]$. Hence, $\Box B$ is false in ωi at $\tau[k]$.

$A = PB$. Suppose that A occurs on b, i.e. $PB, witk$ is on b. Since b is complete (P) has been applied to $PB, witk$. Thus, for some wj, $swiwjtk$ and $B, wjtk$ are on b. By the induction hypothesis, $S\omega i\omega j\tau[k]$ and B is true in ωj at $\tau[k]$. Hence, PB is true in ωi at $\tau[k]$. Suppose $\neg A$ occurs on b, i.e. $\neg PB, witk$ is on b. Since b is complete $(\neg P)$ has been applied to $\neg PB, witk$ and $O\neg B, witk$ is on b, and since b is complete (O) has been applied to $O\neg B, witk$. Thus, for all wj such that $swiwjtk$ is on b, $\neg B, wjtk$ is on b. By the induction hypothesis, for all ωj such that $S\omega i\omega j\tau[k]$, B is false in ωj at $\tau[k]$. Hence, PB is false in ωi at $\tau[k]$.

$A = RtkB$. Suppose that A occurs on b, i.e. $RtkB, witj$ is on b. Then since b is complete (Rt) has been applied to $RtkB, witj$ and $B, witk$ is on b. By the induction hypothesis, B is true in ωi at $\tau[k]$. Hence, $RtkB$ is true in ωi at $\tau[j]$. Suppose that $\neg A$ occurs on b, i.e. that $\neg RtkB, witj$ is on b. Then since b is complete $(\neg Rt)$ has been applied to $\neg RtkB, witj$ and $Rtk\neg B, witj$ is on b, and (Rt) has been applied to $Rtk\neg B, witj$. Thus, $\neg B, witk$ is on b. By the induction hypothesis, B is false in ωi at $\tau[k]$. Hence, $RtkB$ is false in ωi at $\tau[j]$. (This argument goes through for every Rt operator.)

$A = \underline{G}B$. Suppose that A occurs on b, i.e. $\underline{G}B, witj$ is on b. Since b is complete $(\underline{G})$ has been applied to $\underline{G}B, witj$. Thus, for all tk such that $tj < tk$ is on b, $B, witk$ is on b. By the induction hypothesis, for all $\tau[k]$ such that $\tau[j] < \tau[k]$, B is true in ωi at $\tau[k]$. Hence, $\underline{G}B$ is true in ωi at $\tau[j]$. Suppose that $\neg A$ occurs on b, i.e. that $\neg\underline{G}B, witj$ is on b. Since b is complete $(\neg\underline{G})$ has been applied to $\neg\underline{G}B, witj$. Hence, $\underline{F}\neg B, witj$ is on b. Again, since b is complete $(\underline{F})$ has been applied to $\underline{F}\neg B, witj$. So, for some tk, $tj < tk$ and $\neg B, witk$ are on b. By the induction hypothesis, $\tau[j] < \tau[k]$ and B is false in ωi at $\tau[k]$. It follows that $\underline{G}B$ is false in ωi at $\tau[j]$.

Conclusion. The lemma holds for a sentence A of any complexity. □

Theorem 12 *(Completeness theorem). Let S be any of the tableau systems discussed in this essay and let* **M** *be the class of models that corresponds to S. Then S is strongly complete with respect to* **M**.

Proof. The proof is an easy modification of similar proofs in mono-modal logic (see e.g. Priest [33]).

For the weakest system T, the proof goes like this. Suppose that not $\Gamma \vdash_T B$, i.e. it is not the case that there is a closed T-tableau whose initial list comprises $A, w0t0$ for every $A \in \Gamma$ and $\neg B, w0t0$. Let t be a complete T-tableau whose initial list comprises $A, w0t0$ for every $A \in \Gamma$ and $\neg B, w0t0$. Then it is not the case that t is closed, i.e. it is open. Since t is open, there is at least one open branch in t. Let b be an open branch in t. By the Completeness lemma, the model that b induces makes all the premises $A \in \Gamma$ true in ω_0 at $\tau[0]$ and B false in ω_0 at $\tau[0]$. Hence, not **M**, $\Gamma \Vdash B$. Thus, if not $\Gamma \vdash_T B$, then not **M**, $\Gamma \Vdash B$. Consequently, if **M**, $\Gamma \Vdash B$, then $\Gamma \vdash_T B$ (for finite Γ).

In this proof we have assumed that Γ is finite. But we can also prove that the theorem holds when Γ is infinite by adapting the method in Smullyan [37] mentioned above in Section 4.4.

For all other systems, we just have to check that the model induced by the open branch, b, is of the right kind in every case. We only consider some cases to illustrate the method.

$(C\text{-}aT)$ Suppose that $\omega i \in W$. Then wi occurs on b [by Def. IM]. Since b is complete $(T\text{-}aT)$ has been applied. Thus, $rwiwitk$ is on b. Hence, $R\omega i\omega i\tau[k]$, as required [by Def. IM].

$(C\text{-}dT')$ Suppose that $S\omega i\omega j\tau[k]$. Then $swiwjtk$ occurs on b [by Def. IM]. Since the tableau is complete, $(T\text{-}dT')$ has been applied and $swjwjtk$ occurs on b. Hence, $S\omega j\omega j\tau[k]$, as required [by Def. IM].

$(C\text{-}t4)$ Suppose that $\tau[i] < \tau[j]$ and $\tau[j] < \tau[k]$. Then $ti < tj$ and $tj < tk$ occur on b [by Def. IM]. Since b is complete, $(T\text{-}t4)$ has been applied and $ti < tk$ occurs on b. Hence, $\tau[i] < \tau[k]$, as required [by Def. IM].

$(C\text{-}OC)$ Suppose that $\omega i \in W$. Then wi occurs on b [by Def. IM]. Since b is complete $(T\text{-}OC)$ has been applied. Thus, for some wj, $swiwjtk$ and $rwiwjtk$ are on b. Hence, for some ωj, $S\omega i\omega j\tau[k]$ and $R\omega i\omega j\tau[k]$, as required [by Def. IM].

$(C\text{-}MO)$ Suppose that $S\omega i\omega j\tau[k]$. Then $swiwjtk$ occurs on b [by Def. IM]. Since b is complete, $(T\text{-}MO)$ has been applied and $rwiwjtk$ occurs on b. Accordingly, $R\omega i\omega j\tau[k]$, as required [by Def. IM].

$(C\text{-}PMP)$ Suppose that $S\omega i\omega j\tau[m]$ and $R\omega i\omega k\tau[m]$. Then $swiwjtm$ and $rwiwktm$ occur on b [by Def. IM]. Since b is complete, $(T\text{-}PMP)$ has been applied and for some wl, $rwjwltm$ and $swkwltm$ occur on b. Consequently, for some ωl, $R\omega j\omega l\tau[m]$ and $S\omega k\omega l\tau[m]$, as required [by Def. IM].

$(C\text{-}PC)$ Suppose that $\tau[j] < \tau[i]$ and $\tau[k] < \tau[i]$. Then $tj < ti$ and $tk < ti$ are on b. Since b is complete, $(T\text{-}PC)$ has been applied and $tj < tk$ or $tj = tk$ or $tk < tj$ is on b [by Def. IM]. Hence, $\tau[j] < \tau[k]$, $\tau[j] = \tau[k]$ or $\tau[k] < \tau[j]$, as required [by Def. IM]. For if $tj = tk$ is on b, $j \rightleftharpoons k$, and if $j \rightleftharpoons k$, $[j] = [k]$.

$(C\text{-}SP)$ Suppose that $\tau[k] < \tau[l]$ and $R\omega i\omega j\tau[l]$. Then $tk < tl$ and $rwiwjtl$ are on b [by Def. IM]. Since b is complete, $(T\text{-}SP)$ has been applied and $rwiwjtk$ is on b. Hence, $R\omega i\omega j\tau[k]$ as required [by Def. IM].

$(C\text{-}WPI)$ Suppose $\tau[k] < \tau[l]$, $S\omega i\omega j\tau[k]$ and $R\omega i\omega j\tau[l]$. Then $tk < tl$, $swiwjtk$ and $rwiwjtl$ are on b [by Def. IM]. Since b is complete, $(T\text{-}WPI)$ has been applied and $swiwjtl$ is on b. Therefore, $S\omega i\omega j\tau[l]$, as required [by Def. IM].

(C-FT) Suppose that A is true in ωi at $\tau[k]$, where A is atomic, and that $R\omega i\omega j\tau[k]$. Then $A, witk$ and $rwiwjtk$ are on b [by Def. IM]. Since b is complete, (T-FT) has been applied and $A, wjtk$ is on b. Hence, A is true in ωj at $\tau[k]$, as required [by Def. IM]. □

Acknowledgements

We would like to thank Paul Needham, Lennart Åqvist and the anonymous referees for many valuable comments on earlier versions of this paper.

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